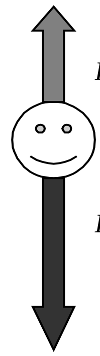
Appendix for Project 2: deriving the velocity formula

Suppose an object with mass *m* is dropped from a certain initial altitude and allowed to fall freely; i.e., the only forces acting on it after its release are gravity and the resistance of the air through which it falls:



First, we need a little physics. The force exerted by gravity equals the mass of the object times the gravitational acceleration constant *g*:

$$F_G = m \times g$$

Assuming constant air density (reasonable if the distance fallen is not too great) and a stable orientation of the object as it falls, the force of air resistance on the object is proportional to the velocity of the object squared:

$$F_D = -k \times v^2$$

where k is a constant that depends on the shape of the object and the density of the air through which it falls and *v* is the velocity of the object. The force is negative since it acts in the opposite direction to the object's motion.

Now, Newton's Second Law of Motion says that the total force acting on an object equals the mass of the object times its acceleration *a*, so we have:

$$F = F_G + F_D \qquad \text{or} \qquad m \times a = m \times g - k \times v^2$$

Now we need a little calculus. The acceleration and velocity are both functions of time *t*, and the acceleration *a* is the derivative of the velocity *v*. So we have:

$$m \times \frac{dv}{dt} = m \times g - k \times v^2 \qquad \text{and also} \qquad v(0) = 0$$

since the object isn't moving when it is released (at time 0). This is an example of a nonlinear differential equation with initial condition (covered in Math 2214). It is possible to solve for a function *v* that satisfies both the differential equation and the initial condition. That process yields the solution:

$$v = \sqrt{\frac{m \times g}{k}} \tanh\left(\sqrt{\frac{g \times k}{m}} \times t \right)$$