Appendix for Project 2: deriving Newton's Law of Cooling

Suppose that an object has an initial temperature of T_0, and that at time $t = 0$ the object is placed into a surrounding medium which is at a constant temperature α. Now, the rate of change of the temperature of the object is proportional to the difference between the temperature of the object and the temperature of the surrounding medium.

Now we need a little calculus. Mathematically, the rate of change is given by the derivative, so we have a differential equation with an initial value condition:

$$\frac{dT}{dt} = -k \times (T - \alpha) \quad \text{and} \quad T(0) = T_0$$

where k is some, as yet unknown, positive constant. (Remember, the object is cooling, so its temperature is decreasing, so the rate of change of the temperature is negative. It is possible to solve for a function $T(t)$ that satisfies both the differential equation and the initial condition.

To start, express in differential notation and do some algebra to separate the variables:

$$\frac{dT}{T - \alpha} = -kdt$$

Now antidifferentiate each side of the equation. This is straight-forward if you know integral calculus:

$$\int \frac{dT}{T - \alpha} = \int -kdt$$

This yields:

$$\ln|T - \alpha| = -kt + C$$

Since the object is cooling, we know $T - \alpha > 0$ so we can drop the absolute value. Exponentiate each side to get:

$$e^{\ln(T-\alpha)} = e^{-kt+C}$$

$$T - \alpha = e^{-kt+C} = e^C e^{-kt} = Ae^{-kt}$$

where $A = e^C$ is a positive constant. So we have:

$$T = \alpha + Ae^{-kt}$$

Now letting $t = 0$, and using the initial condition stated previously we get $A = T_0 - a$ and so:

$$T = \alpha + (T_0 - \alpha)e^{-kt}$$

Now, the rate of cooling would be the absolute value of the derivative of T. The derivative is:

$$T' = -k(T_0 - \alpha)e^{-kt}$$

and so the rate of cooling would be

$$R = k\left(T_0 - \alpha\right)e^{-kt}$$